Vector Polarization Induced by a Viscous Gas Flow in a Magnetic Field

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It is shown that a viscous flow of a polyatomic gas in a magnetic field can give rise to a vector polarization of the molecular rotational angular momenta. This viscomagnetic vector polarization is calculated from an appropriate set of moment equations pertaining to the linearized Waldmann-Snider equation. In particular, the transverse effect is discussed and an estimate of its order of magnitude is given for nitrogen.

Nonequilibrium alignment phenomena in dilute gases of polyatomic molecules or atomic vapors have been subject of several papers in the last decade [1–3]. In particular, in a viscous streaming gas with a velocity gradient $\overline{\nabla \boldsymbol{v}}$ a tensor polarization $\langle \overline{\boldsymbol{J}\boldsymbol{J}} \rangle$ of the molecular rotational angular momenta $\hbar \boldsymbol{J}$ ($\langle \ldots \rangle$ denotes an average over the nonequilibrium distribution function) is produced by collisions if the molecular interaction is nonspherical. The tensor polarization gives rise to an anisotropic part of the dielectric tensor if the electric polarizability of the molecule is anisotropic. This phenomenon of flow birefringence has been studied extensively, both theoretically [4] and experimentally [5]. Indirectly related to this alignment is the Senftleben-Beenakker effect of the viscosity [1, 6]: Since the tensor polarization, which influences the momentum transport, precesses in an external magnetic field, the viscosity coefficients become field dependent.

The simplest case of an alignment, however, is the vector polarization $\langle \boldsymbol{J} \rangle$. But without an external field it cannot exist in a viscous flowing gas or a heat conducting gas. If a temperature gradient is present, one has the constitutive law

$$\langle J_\mu \rangle = \alpha_{\mu\nu} \nabla_\nu T ,$$

where $\alpha_{\mu\nu}$ has to be a pseudo-tensor since parity is conserved in molecular interaction. This tensor $\alpha_{\mu\nu}$ can be constructed from an electric field $\boldsymbol{E}$ (which is a polar vector). Because of parity reasons the only possibility is (with $\varepsilon_{\mu\lambda\nu}$ being the totally antisymmetric isotropic third rank tensor)

$$\alpha_{\mu\nu} = \alpha\, \varepsilon_{\mu\lambda\nu} E_\lambda$$

which gives rise to a vector polarization (in a gas of polar molecules) perpendicular to electric field and temperature gradient:

$$\langle \boldsymbol{J} \rangle = \alpha\, \boldsymbol{E} \times \boldsymbol{\nabla} T .$$

The coefficient α has been related to kinetic theory collision integrals by Hamer and Knaap [7].

If a velocity gradient is present, the vector polarization and $\overline{\nabla \boldsymbol{v}}$ are connected by a third rank true tensor

$$\langle J_\mu \rangle = \beta_{\mu\nu\lambda} \overline{\nabla_\nu v_\lambda} . \tag{1}$$

In an isotropic gas, the tensor $\beta_{\mu\nu\lambda}$ must be proportional to an isotropic tensor of third rank. The only available one is the totally antisymmetric tensor $\varepsilon_{\mu\nu\lambda}$. Since $\varepsilon_{\mu\nu\lambda} \overline{\nabla_\nu v_\lambda} \equiv 0$, a vector polarization cannot be produced directly. [It should be noted, however, that a vector polarization due to a rotation of the velocity can exist (Barnett-effect) without a field:

$$\langle \boldsymbol{J} \rangle = \alpha_{\mathrm{B}} \operatorname{rot} \boldsymbol{v} .$$

The Barnett coefficient α_{B} usually is very small [8] so that the local Barnett polarization (which would occur in a Couette- or Poiseuille flow) can be disregarded.] If, however, a magnetic field $\boldsymbol{H} = H \boldsymbol{h}$ is present, $\beta_{\mu\nu\lambda}$ also contains parts symmetric with respect to ν and λ:

$$\beta_{\mu\nu\lambda} = \beta_1(H)\,(h_\nu \delta_{\mu\lambda} + h_\lambda \delta_{\mu\nu}) + \beta_2(H) \times (\varepsilon_{\mu\varkappa\lambda} h_\varkappa h_\nu + \varepsilon_{\mu\varkappa\nu} h_\varkappa h_\lambda) + \beta_3(H)\, h_\mu h_\nu h_\lambda . \tag{2}$$

It is the task of the kinetic theory of polyatomic gases based on the Waldmann-Snider equation [9, 10], to derive Eqs. (1, 2) and to find expressions for the coefficients β_1, β_2, β_3 in terms of properties of

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single molecules (i. e. magnetic moment) and their mutual nonspherical interaction.

The paper proceeds as follows: From the linearized Waldmann-Snider equation [8–12], a set of coupled tensor equations for the appropriate macroscopic observables (moments) is derived. By use of the projection operator technique developed by Hess and Waldmann [13] we give an expression for $\langle \boldsymbol{J} \rangle$ in terms of $\overline{\nabla \boldsymbol{v}}$ and magnetic field $\boldsymbol{H}$. The proportionality coefficient contains certain Waldmann-Snider collision integrals [11, 14], whose connection with the nonspherical molecular interaction is briefly discussed. A possible experimental device for the measurement of the effect is described; the order of magnitude of expected results is estimated.

1. Moment Equations

In this section the transport-relaxation equations needed for the treatment of the flow induced vector polarization in a magnetic field are derived. The starting point is the Waldmann-Snider equation [9, 10] for the distribution operator $f(t, \boldsymbol{x}, \boldsymbol{c}, \boldsymbol{J})$ of the gas ($\boldsymbol{c}$ is the molecular velocity and $\hbar \boldsymbol{J}$ is the rotational angular momentum of a molecule). For simplicity, restriction is made to linear diamagnetic molecules with an internal Hamiltonian $\mathcal{H}_{\text{rot}} = \hbar^2 J^2/2\,\Theta$ where Θ is the moment of inertia.

For the nonequilibrium distribution f the usual ansatz

$$f(t, \boldsymbol{x}, \boldsymbol{c}, \boldsymbol{J}) = f_0(c^2, J^2)[1 + \Phi(t, \boldsymbol{x}, \boldsymbol{c}, \boldsymbol{J})] \quad (3)$$

is made. In Eq. (3)

$$f_0 = n_0\, Q_{\text{rot}}^{-1} (m/2\,\pi\, k_B T_0)^{3/2} \exp\left(-\frac{m\,c^2}{2\,k_B T_0} - \frac{\mathcal{H}_{\text{rot}}}{k_B T_0}\right) \quad (4)$$

is a global Maxwellian with temperature T_0 and particle number density n_0; $Q_{\text{rot}} = \operatorname{Tr} \mathcal{H}_{\text{rot}}$ is the rotational partition function, "Tr" stands for a sum over rotational quantum numbers and the trace over magnetic quantum numbers. The relative deviation of the distribution from equilibrium, Φ, obeys the linearized Waldmann-Snider equation

$$\frac{\partial \Phi}{\partial t} + \boldsymbol{c} \cdot \nabla \Phi - i\,\omega_L [\boldsymbol{h} \cdot \boldsymbol{J}, \Phi]_- + \omega(\Phi) = 0\,. \quad (5)$$

Here, $\omega_L = \mu_n\, g_n\, H/\hbar$ is the Larmor frequency (μ_n being the nuclear magneton, g_n the rotational g-factor, H the magnitude of the magnetic field), $\boldsymbol{h}$ is a unit vector in the direction of the magnetic field and $\omega(\Phi)$ is the linearized Waldmann-Snider collision operator [8–12] which contains the binary scattering amplitude matrix and its adjoint.

In the moment method [8, 12], the linearized W-S equation is transformed into an (infinite) set of coupled differential equations for the moments of the distribution function. For special problems only a small finite subset of expansion tensors (the averages of which are the moments) is needed. For our case these expansion tensors are given in Table 1.

Table 1. Orthonormalized expansion tensors used for the treatment of flow induced vector polarization.

Vectors	Second rank tensor	Third rank tensor
$\Phi_\mu^{(10)} = \sqrt{2}\, W_\mu$	$\Phi_{\mu\nu}^{(20)} = \sqrt{2}\, \overline{W_\mu W_\nu}$	$\Phi_{\mu\nu,\lambda}^{(21)} = \sqrt{\frac{6}{\langle J^2 \rangle_0}} \cdot \overline{W_\mu W_\nu} J_\lambda$
$\Phi_\mu^{(01)} = \sqrt{\frac{3}{\langle J^2 \rangle_0}}\, J_\mu$		

Here, $\boldsymbol{W} = (m/2\,k_B T_0)^{1/2}\, \boldsymbol{c}$ is the dimensionless molecular velocity, the bar $\overline{}$ denotes the irreducible part of a tensor and the upper indices indicate the tensor ranks in $\boldsymbol{W}$ and $\boldsymbol{J}$, respectively. Notice, that $\Phi_{\mu\nu,\lambda}^{(21)}$ is not irreducible. The expansion tensors are mutually orthogonal and normalized according to

$$\langle \Phi_\mu^{(10)} \Phi_{\mu'}^{(10)} \rangle_0 = \langle \Phi_\mu^{(01)} \Phi_{\mu'}^{(01)} \rangle_0 = \delta_{\mu\mu'}\,, \quad (6\,\text{a})$$

$$\langle \Phi_{\mu\nu}^{(20)} \Phi_{\mu'\nu'}^{(20)} \rangle_0 = \Delta_{\mu\nu,\mu'\nu'}\,, \quad (6\,\text{b})$$

$$\langle \Phi_{\mu\nu,\lambda}^{(21)} \Phi_{\mu'\nu',\lambda'}^{(21)} \rangle_0 = \delta_{\lambda\lambda'}\, \Delta_{\mu\nu,\mu'\nu'}\,. \quad (6\,\text{c})$$

The symbol $\langle \ldots \rangle_0$ denotes an equilibrium average and $\Delta_{\mu\nu,\mu'\nu'} \equiv \frac{1}{2}(\delta_{\mu\mu'}\,\delta_{\nu\nu'} + \delta_{\mu\nu'}\,\delta_{\nu\mu'}) - \frac{1}{3}\,\delta_{\mu\nu}\,\delta_{\mu'\nu'}$ is a fourth rank tensor which projects out the irreducible part if applied to an arbitrary second rank tensor. Introducing $a_\mu \equiv \langle \Phi_\mu^{(10)} \rangle$ (mean velocity), $b_\mu \equiv \langle \Phi_\mu^{(01)} \rangle$ (vector polarization), $a_{\mu\nu} \equiv \langle \Phi_{\mu\nu}^{(20)} \rangle$ (friction pressure tensor) and $a_{\mu\nu,\lambda} \equiv \langle \Phi_{\mu\nu,\lambda}^{(21)} \rangle$, the correction function Φ can be written as

$$\Phi = a_\mu\, \Phi_\mu^{(10)} + b_\mu\, \Phi_\mu^{(01)} + a_{\mu\nu}\, \Phi_{\mu\nu}^{(20)} + a_{\mu\nu,\lambda}\, \Phi_{\mu\nu,\lambda}^{(21)}\,. \quad (7)$$

Now, the expansion Eq. (7) is inserted into Eq. (5), the resulting equation is multiplied with $\Phi_\mu^{(01)}$, $\Phi_{\mu\nu}^{(20)}$ and $\Phi_{\mu\nu,\lambda}^{(21)}$, respectively, and the equilibrium average is taken. (The equation for $\Phi_\mu^{(10)}$ is not needed, it gives only the local conservation of momentum.) This results in the three coupled moment equations, taken for a stationary situation:

$$\sqrt{2}\,\overline{\nabla_\mu v_\nu} + \langle \Phi^{(20)}_{\mu\nu}\, \omega(\Phi^{(20)}_{\mu'\nu'}) \rangle_0\, a_{\mu'\nu'} + \langle \Phi^{(20)}_{\mu\nu}\, \omega(\Phi^{(21)}_{\mu'\nu',\lambda'}) \rangle_0\, a_{\mu'\nu',\lambda'} = 0\,, \quad (8)$$

$$\omega_\mathrm{L}(\boldsymbol{h}\times\boldsymbol{b})_\mu + \langle \Phi^{(01)}_{\mu}\, \omega(\Phi^{(01)}_{\mu'}) \rangle_0\, b_{\mu'} + \langle \Phi^{(01)}_{\mu}\, \omega(\Phi^{(21)}_{\mu'\nu',\lambda'}) \rangle_0\, a_{\mu'\nu',\lambda'} = 0\,, \quad (9)$$

$$\omega_\mathrm{L}\, \varepsilon_{\lambda\varrho\lambda'}\, h_\varrho\, a_{\mu\nu,\lambda'} + \langle \Phi^{(21)}_{\mu\nu,\lambda}\, \omega(\Phi^{(21)}_{\mu'\nu',\lambda'}) \rangle_0\, a_{\mu'\nu',\lambda'} + \langle \Phi^{(21)}_{\mu\nu,\lambda}\, \omega(\Phi^{(01)}_{\mu'}) \rangle_0\, b_{\mu'} + \langle \Phi^{(21)}_{\mu\nu,\lambda}\, \omega(\Phi^{(20)}_{\mu'\nu'}) \rangle_0\, a_{\mu'\nu'} = 0\,. \quad (10)$$

Due to the rotational invariance of the collision operator the Wigner-Eckart Theorem can be applied to the collision brackets yielding

$$\langle \Phi^{(01)}_{\mu}\, \omega(\Phi^{(01)}_{\mu'}) \rangle_0 = n_0\, v_\mathrm{rel}\, \mathfrak{S}\binom{01}{01}\, \delta_{\mu\mu'}\,, \quad (11)$$

$$\langle \Phi^{(20)}_{\mu\nu}\, \omega(\Phi^{(20)}_{\mu'\nu'}) \rangle_0 = n_0\, v_\mathrm{rel}\, \mathfrak{S}\binom{20}{20}\, \Delta_{\mu\nu,\mu'\nu'}\,, \quad (12)$$

$$\langle \Phi^{(01)}_{\mu}\, \omega(\Phi^{(21)}_{\mu'\nu',\lambda'}) \rangle_0 = \sqrt{\tfrac{3}{5}}\, n_0\, v_\mathrm{rel}\, \mathfrak{S}\binom{01}{21}\, \Delta_{\mu\lambda',\mu'\nu'}\,, \quad (13)$$

$$\langle \Phi^{(20)}_{\mu\nu}\, \omega(\Phi^{(21)}_{\mu'\nu',\lambda'}) \rangle_0 = \sqrt{\tfrac{2}{5}}\, n_0\, v_\mathrm{rel}\, \mathfrak{S}\binom{20}{21}\, \square_{\mu\nu,\lambda',\mu'\nu'}\,, \quad (14)$$

and

$$\langle \Phi^{(20)}_{\mu\nu,\lambda}\, \omega(\Phi^{(21)}_{\mu'\nu',\lambda'}) \rangle_0 \approx n_0\, v_\mathrm{rel}\, \mathfrak{S}\binom{21}{21}\, \delta_{\lambda\lambda'}\, \Delta_{\mu\nu,\mu'\nu'}\,. \quad (15)$$

The $\mathfrak{S}\binom{p\ q}{p'\ q'}$ are effective cross sections $[\mathfrak{S}\binom{pq}{pq} > 0]$ being only dependent on temperature, $v_\mathrm{rel} = 4\,(k_\mathrm{B}\,T_0/\pi\, m)^{1/2}$ is a thermal velocity. While the first four relations, Eqs. (11) – (14) are exact, Eq. (15) is only valid approximately ("spherical approximation", cf. Refs. [15, 11]) since the Wigner-Eckart Theorem can be applied only to totally irreducible tensors. But this approximation works well if the nonspherical part of the molecular interaction is small compared to the spherical part and seems to be applicable also for the more general case [1]. The tensor $\square_{\mu\nu,\lambda',\mu'\nu'}$ occurring in Eq. (14) is defined by

$$\square_{\mu\nu,\lambda',\mu'\nu'} = \tfrac{1}{4}(\varepsilon_{\mu\lambda'\mu'}\,\delta_{\nu\nu'} + \varepsilon_{\mu\lambda'\nu'}\,\delta_{\nu\mu'} + \varepsilon_{\nu\lambda'\mu'}\,\delta_{\mu\nu'} + \varepsilon_{\nu\lambda\nu'}\,\delta_{\mu\mu'})\,.$$

Because of time reversal invariance of the molecular interaction the Onsager symmetries $\mathfrak{S}\binom{01}{21} = \mathfrak{S}\binom{21}{01}$ and $\mathfrak{S}\binom{20}{21} = -\mathfrak{S}\binom{21}{20}$ hold [8]. Equations (8) – (10) can now be rewritten in the form

$$a_{\mu\nu} - \sqrt{\frac{2}{5}}\,\frac{\mathfrak{S}\binom{21}{20}}{\mathfrak{S}\binom{20}{20}}\,\square_{\mu\nu,\lambda',\mu'\nu'}\, a_{\mu'\nu',\lambda'} = \frac{-\sqrt{2}}{n_0\, v_\mathrm{rel}\, \mathfrak{S}\binom{20}{20}}\,\overline{\nabla_\nu v_\mu}\,, \quad (16)$$

$$b_\mu + \varphi_{01}\,(\boldsymbol{h}\times\boldsymbol{b})_\mu + \sqrt{\frac{3}{5}}\,\frac{\mathfrak{S}\binom{21}{01}}{\mathfrak{S}\binom{01}{01}}\, a_{\mu\lambda,\lambda} = 0\,, \quad (17)$$

$$a_{\mu\nu,\lambda} + \varphi_{21}\,(\boldsymbol{h}\times\mathbf{a})_{\mu\nu,\lambda} + \sqrt{\frac{2}{5}}\,\frac{\mathfrak{S}\binom{21}{20}}{\mathfrak{S}\binom{21}{21}}\,\square_{\mu\nu,\lambda,\mu'\nu'}\, a_{\mu'\nu'} + \sqrt{\frac{3}{5}}\,\frac{\mathfrak{S}\binom{21}{01}}{\mathfrak{S}\binom{21}{21}}\,\Delta_{\mu\nu,\lambda\mu'}\, b_{\mu'} = 0\,. \quad (18)$$

Here, the effective precession angles

$$\varphi_{01} \equiv \omega_\mathrm{L}/n_0\, v_\mathrm{rel}\, \mathfrak{S}\binom{01}{01} \quad \text{and} \quad \varphi_{21} \equiv \omega_\mathrm{L}/n_0\, v_\mathrm{rel}\, \mathfrak{S}\binom{21}{21}$$

have been introduced. The Eqs. (16) – (18) now have to be solved to find $\boldsymbol{b}$ in terms of $\overline{\boldsymbol{\nabla v}}$ and the magnetic field.

2. The Viscomagnetic Vector Polarization

To obtain an expression for the flow induced vector polarization $\boldsymbol{b}$ in a magnetic field, the tensors $a_{\mu\nu}$ and $a_{\mu\nu,\lambda}$ have to be eliminited in Equations (16) – (18). Since we are interested only in the lowest-order-in-nonsphericity contributions, we can neglect the second term in Eq. (16); thus

$$a_{\mu\nu} \approx -(\sqrt{2}/[n_0\, v_\mathrm{rel}\, \mathfrak{S}\binom{20}{20}]\,\overline{\nabla_\nu v_\mu}\,. \quad (19)$$

Next, we express b_μ in terms of $a_{\mu\lambda,\lambda}$ and then $a_{\mu\nu,\lambda}$ in terms of $a_{\mu\nu}$. For this purpose, the projection operator method developed by Hess and Waldmann [13] is used: Projection operators, acting on vectors, are introduced by

$$P^{(0)}_{\mu\nu} \equiv h_\mu\, h_\nu\,,\; P^{(\pm 1)}_{\mu\nu} \equiv \tfrac{1}{2}(\delta_{\mu\nu} - h_\mu\, h_\nu \mp i\, \varepsilon_{\mu\lambda\nu}\, h_\lambda)\,. \quad (20)$$

They have the properties

$$P^{(m)}_{\mu\nu}\, P^{(m')}_{\nu\lambda} = \delta^{mm'}\, P^{(m)}_{\mu\lambda}\,, \quad \sum_{m=0,\pm 1} P^{(m)}_{\mu\nu} = \delta_{\mu\nu}\,. \quad (21)$$

An equation of the form

$$\sum_m \alpha_m\, P^{(m)}_{\mu\nu}\, A_\nu = B_\mu$$

is, with the help of Eqs. (21) immediately solved to give

$$A_\mu = \sum_m (\alpha_m)^{-1}\, P^{(m)}_{\mu\nu}\, B_\nu\,.$$

Since $\boldsymbol{h}\times\boldsymbol{A} = i(\mathbf{P}^{(+1)} - \mathbf{P}^{(-1)})\cdot\boldsymbol{A}$ the solution for $\boldsymbol{b}$ in Eq. (17) is found as

$$b_\mu = -\sqrt{\frac{3}{5}}\,\frac{\mathfrak{S}\binom{21}{01}}{\mathfrak{S}\binom{01}{01}}\,[(1 + i\,\varphi_{01})^{-1}\, P^{(+1)}_{\mu\nu} + (1 - i\,\varphi_{01})^{-1}\, P^{(-1)}_{\mu\nu} + P^{(0)}_{\mu\nu}]\, a_{\nu\lambda,\lambda}\,. \quad (22)$$

In Eq. (18) we can neglect the term proportional to $\boldsymbol{b}$ since it would only contribute in higher order in nondiagonal collision integrals. Analogously we find

$$a_{\mu\nu,\lambda} = -\sqrt{\frac{2}{5}\frac{\mathfrak{S}\binom{21}{20}}{\mathfrak{S}\binom{21}{21}}} \cdot [(1+i\varphi_{21})^{-1} P^{(+1)}_{\lambda\varkappa} + (1-i\varphi_{21})^{-1} P^{(-1)}_{\lambda\varkappa} + P^{(0)}_{\lambda\varkappa}] \cdot \Box_{\mu\nu,\varkappa,\varrho\tau}\, a_{\varrho\tau}\,. \tag{23}$$

Thus, finally b_μ is obtained as

$$b_\mu = \frac{\sqrt{6}}{5}\frac{\mathfrak{S}\binom{21}{01}\mathfrak{S}\binom{21}{20}}{\mathfrak{S}\binom{01}{01}\mathfrak{S}\binom{21}{21}} \cdot [(1+i\varphi_{01})^{-1} P^{(1)}_{\mu\nu} + (1-i\varphi_{01})^{-1} P^{(-1)}_{\mu\nu} + P^{(0)}_{\mu\nu}] \cdot [(1+i\varphi_{21})^{-1} P^{(1)}_{\lambda\varkappa} + (1-i\varphi_{21})^{-1} P^{(-1)}_{\lambda\varkappa} + P^{(0)}_{\lambda\varkappa}] \cdot \Box_{\lambda\nu,\varkappa,\varrho\tau}\, a_{\varrho\tau}\,. \tag{24}$$

Since for zero magnetic field the brackets give $\delta_{\mu\nu}\,\delta_{\lambda\varkappa}$ and since $\Box_{\varkappa\mu,\varkappa,\varrho\tau} \equiv 0$ there is no effect without a magnetic field, as was expected.

Using the constraint $\operatorname{div} \boldsymbol{v} = 0$ and the relation $h_\mu\, h_\nu\, h_\lambda\, h_\varkappa\, \Box_{\lambda\nu,\varkappa,\varrho\tau} \equiv 0$ we obtain, after some labour, an expression for the viscomagnetic vector polarization

$$\langle \boldsymbol{J} \rangle = -\frac{1}{5}\langle J^2\rangle_0^{1/2}\frac{\eta}{p}\frac{\mathfrak{S}\binom{21}{01}\mathfrak{S}\binom{21}{20}}{\mathfrak{S}\binom{01}{01}\mathfrak{S}\binom{21}{21}} \cdot \frac{\varphi_{21}}{(1+\varphi_{21}^2)(1+\varphi_{01}^2)}\{(3+\varphi_{01}\varphi_{21})\boldsymbol{h}\cdot\overline{\boldsymbol{\nabla}\boldsymbol{v}} + (\varphi_{21}-3\varphi_{01})\,\boldsymbol{h}\times(\boldsymbol{h}\cdot\overline{\boldsymbol{\nabla}\boldsymbol{v}}) + \varphi_{01}(3\varphi_{01}-\varphi_{21}) \cdot \boldsymbol{h}\,(\boldsymbol{h}\cdot\overline{\boldsymbol{\nabla}\boldsymbol{v}}\cdot\boldsymbol{h})\}\,. \tag{25}$$

Here, we have introduced the viscosity η by $\eta = p/n_0\, v_{\text{rel}}\, \mathfrak{S}\binom{20}{20}$. The effect is inversely proportional to the pressure p. It approaches zero for high magnetic fields due to the destruction of the polarization by the fast precession. The coefficients β_1, β_2, β_3 introduced phenomenologically [see Eq. (2)] can immediately be inferred from Equation (25).

Next, some remarks on the relevant effective cross sections are in order. We use the general form of an effective cross section, presented in Ref. [11] and introduce for convenience the following bracket symbol:

$$[(\ldots)] \equiv 2\pi\, Q_{\text{rot}}^{-2} \sum_{j_1 j_2 j_1' j_2'} \operatorname{tr}_1 \operatorname{tr}_2 \iint \mathrm{d}\gamma\, \mathrm{d}\vartheta \cdot \exp\{-\gamma^2 - \varepsilon(j_1) - \varepsilon(j_2)\}\, \gamma^2\, \gamma'\, (\ldots)_{j_1 j_2 j_1' j_2'} \sin\vartheta\,. \tag{26}$$

Here, j_1', j_2', $\boldsymbol{\gamma}' = \gamma'\,\boldsymbol{e}'$ and j_1, j_2, $\boldsymbol{\gamma} = \gamma\,\boldsymbol{e}$ are the rotational quantum numbers and dimensionless relative velocities before and after the collision, respectively; $\varepsilon(j)$ is the rotational energy divided by $k_B T_0$; "tr" denotes the trace over magnetic quantum numbers and $\vartheta = \arccos(\boldsymbol{e}\cdot\boldsymbol{e}')$ is the angle of deflection in the c.m. system. With the use of results of Ref. [11] (optical theorem for the scattering amplitude) the following expressions for the effective cross sections are found:

$$\mathfrak{S}\binom{01}{01} = 2\,\langle J^2\rangle_0^{-1}[\boldsymbol{J}_1\, a\cdot[a^+, \boldsymbol{J}_1+\boldsymbol{J}_2]_-]\,, \tag{27}$$

$$\mathfrak{S}\binom{21}{01} = \sqrt{\tfrac{6}{5}}\,\langle J^2\rangle_0^{-1}[\gamma^2\,\overline{\boldsymbol{e}\boldsymbol{e}} : \boldsymbol{J}_1\, a\,[a^+, \boldsymbol{J}_1+\boldsymbol{J}_2]_-]\,, \tag{28}$$

$$\mathfrak{S}\binom{21}{20} = -\sqrt{\tfrac{2}{15}}\,\langle J^2\rangle_0^{-1/2}[\gamma^2\,\gamma'^2\cos\vartheta \cdot (a\, a^+ \boldsymbol{J}_1)\cdot(\boldsymbol{e}'\times\boldsymbol{e})]\,. \tag{29}$$

In Eqs. (27) – (29), $a \equiv a^{j_1 j_2 j_1' j_2'}(E, \boldsymbol{e}, \boldsymbol{e}')$ is the binary scattering amplitude matrix. The above cross sections are essentially connected with the nonspherical part of the interaction since they vanish for a purely spherical potential where the pertaining scattering amplitude a_{sph} commutes with $\boldsymbol{J}_1$, $\boldsymbol{J}_2$. For an illustration of Eqs. (27) – (29) we consider a (somewhat unrealistic) elastic "nonspherical" scattering amplitude of the simple form $a = a_0(E, \vartheta) + a_1(E, \vartheta)\,(\boldsymbol{J}_1+\boldsymbol{J}_2)\cdot(\boldsymbol{e}'\times\boldsymbol{e})/\sin\vartheta$. Then we obtain:

$$\mathfrak{S}\binom{01}{01} = \frac{4\pi}{3}\int_0^\infty \mathrm{d}\gamma \int_0^\pi \sin\vartheta\, \mathrm{d}\vartheta\, \exp(-\gamma^2)\,\gamma^3\, |a_1|^2\,,$$

$$\mathfrak{S}\binom{21}{01} = \frac{\pi}{9}\sqrt{\frac{6}{5}}\int_0^\infty \mathrm{d}\gamma \int_0^\pi \sin\vartheta\, \mathrm{d}\vartheta\, \exp(-\gamma^2)\,\gamma^5\, |a_1|^2\,,$$

$$\mathfrak{S}\binom{21}{20} = -\frac{\pi}{3}\sqrt{\frac{2}{15}}\,\langle J^2\rangle_0^{1/2}\int_0^\infty \mathrm{d}\gamma \int_0^\pi \sin\vartheta\, \mathrm{d}\vartheta \cdot \exp(-\gamma^2)\,\gamma^7 \sin 2\vartheta\, \operatorname{Re}(a_0^*\, a_1)\,.$$

For the two relaxation cross sections $\mathfrak{S}\binom{20}{20}$ and $\mathfrak{S}\binom{21}{21}$, the "spherical approximation" [11] yields

$$\mathfrak{S}\binom{21}{21} \approx \mathfrak{S}\binom{20}{20} \approx \tfrac{8}{5}\, v_{\text{rel}}^{-1}\, \Omega^{(2,2)} \tag{30}$$

where Ω is a well-known Chapman-Cowling integral [16]. In a first order DWBA one has $a = a_{\text{sph}} + \varepsilon\, a_{\text{nonsph}}$, where ε is a rough measure for the ratio of the nonspherical and the spherical part of the potential and where a_{nonsph} is linear in the nonspherical part of the interaction [11]. It can be shown, that $\mathfrak{S}\binom{01}{01}$, $\mathfrak{S}\binom{21}{01}$ and $\mathfrak{S}\binom{21}{20}$ are of order ε^2 for linear molecules [11]. Thus the whole effect is of order ε^2, a result which is also true for the Senftleben-Beenakker effect on the viscosity [6].

For a comparison, the expression for the flow induced tensor polarization $\langle \overline{\boldsymbol{J}\boldsymbol{J}} \rangle$ is given [4]:

$$\langle \overline{\boldsymbol{J}\boldsymbol{J}} \rangle = \frac{2}{\sqrt{15}}\langle J^2(J^2-\tfrac{3}{4})\rangle_0^{1/2}\frac{\eta}{p}\frac{\mathfrak{S}\binom{02}{20}}{\mathfrak{S}\binom{02}{02}}\overline{\boldsymbol{\nabla}\boldsymbol{v}} \tag{31}$$

which is, in first order DWBA, independent of the nonsphericity parameter [11]. Thus, for the hydrogen molecules the flow induced vector polarization is certainly much smaller than the tensor polarization. But for molecules with large nonsphericity a comparison on the basis of ε cannot be made.

3. Discussion: Estimate of the Order of Magnitude

Consider a polyatomic gas flowing in x-direction through a rectangular channel with the dimensions $L_x, L_y, L_z (L_x, L_y \gg L_z)$. Then, approximately one has $\overline{\nabla \boldsymbol{v}} \approx (\partial v/\partial z)\, \overline{\boldsymbol{e}_x \boldsymbol{e}_z}$. The magnetic field is in z-direction and we are looking for the vector polarization in y-direction. For convenience a scale factor $a \equiv \mathfrak{S}(\substack{21\\21})/\mathfrak{S}(\substack{01\\01})$ is introduced. Furthermore, $\langle J^2 \rangle_0$ is evaluated in the high temperature limit yielding $\langle J^2 \rangle_0 \approx T/T_{\text{rot}}$ where $T_{\text{rot}} = \hbar^2/2\, k_{\text{B}}\, \Theta$. For the transverse vector polarization $\langle J_y \rangle$ we then obtain:

$$\langle J_y \rangle \approx \frac{a(3a-1)}{10} \frac{\partial v}{\partial z} \frac{\eta}{p} \left(\frac{T}{T_{\text{rot}}}\right)^{1/2} \frac{\mathfrak{S}(\substack{01\\21})\, \mathfrak{S}(\substack{21\\20})}{\mathfrak{S}(\substack{21\\21})^2} \cdot \frac{\varphi_{21}^2}{(1+\varphi_{21}^2)(1+a^2 \varphi_{21}^2)} . \qquad (32)$$

The effect is maximal for $(\varphi_{21})_{\max} = a^{-1/2}$.

Next, we give an estimate of the order of magnitude of the maximal effect for nitrogen at $T = 300$ K, $p = 1$ Torr. We take $a = \mathfrak{S}(\substack{21\\21})/\mathfrak{S}(\substack{01\\01}) \sim 2$ [which is the order of the known ratio $\mathfrak{S}(\substack{12\\12})/\mathfrak{S}(\substack{02\\02})$]. Then $(\varphi_{21})_{\max}$ corresponds to a magnetic field of about 1 kG. The velocity gradient near the boundary is $\partial v/\partial z \sim L_z\, \delta p / L_x\, \eta$ where δp is the pressure difference along the x-direction. With

$$L_z/L_x \sim 10^{-1}, \quad T_{\text{rot}} = 2.9\ \text{K}, \quad \delta p/p \sim 0.5 \quad \text{and} \quad \mathfrak{S}(\substack{01\\21})\, \mathfrak{S}(\substack{21\\20})/\mathfrak{S}(\substack{21\\21})^2 \sim 10^{-3}$$

one finds $\langle J_y \rangle \sim 5 \cdot 10^{-5}$. This corresponds to a purely directional alignment $\langle J_y \rangle / \sqrt{\langle J^2 \rangle_0}$ of the order $5 \cdot 10^{-6}$. For comparison, the magnitude of the corresponding component of the tensor polarization $\langle \overline{J_z J_x} \rangle / \sqrt{\langle J^2 (J^2 - \frac{3}{4}) \rangle_0}$ at the same pressure is estimated:

$$\frac{\langle \overline{J_z J_x} \rangle}{\sqrt{\langle J^2 (J^2 - \frac{3}{4}) \rangle_0}} \approx \frac{1}{\sqrt{15}} \frac{L_z\, \delta p}{L_x\, p} \frac{\mathfrak{S}(\substack{02\\20})}{\mathfrak{S}(\substack{02\\02})} \sim 10^{-4} ,$$

which is at least one order of magnitude larger.

On the other hand, the (longitudinal) equilibrium vector polarization $\langle J_z \rangle_0 \approx g_n \mu_n H \langle J^2 \rangle_0 / 3\, k_{\text{B}}\, T_0$ is, for $H = 1$ kG, of the order 10^{-6}, i.e. at least one order of magnitude smaller than $\langle J_y \rangle$.

The viscomagnetic vector polarization should be still of detectable size. It gives rise to a macroscopic magnetization of the gas

$$\boldsymbol{M} = n_0\, \mu_n\, g_n \langle \boldsymbol{J} \rangle \qquad (33)$$

which is pressure independent. The transverse component M_y should be measurable if a time varying magnetic field H is used.

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